Representation Varieties for Upper Triangular Matrices

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Introduction

M = closed connected manifold

$\pi_1(M)$ = fundamental group

G = algebraic group over k

G-representation variety of M $\qquad R_G(M) = \mathrm{Hom}(\pi_1(M), G)$

$$R_G(\Sigma_g) = \left\{ (A_1, B_1, \ldots, A_g, B_g) \in G^{2g} : \prod_{i=1}^{g} [A_i, B_i] = 1 \right\}$$

History

- **Morse theory** Poincaré polynomials of SL_2, SL_3, GL_4-character varieties

  (Hitchin, Gothen, García-Prada, Heinloth, Schmidt, ...)

- **Arithmetic method** E-polynomial of (twisted) SL_n, GL_n-character varieties

  (Hausel, Rodríguez-Villegas, Mereb, ...)

- **Geometric method** E-polynomial of (untwisted) SL_2, PGL_2-character varieties

  (Logares, Martinez, Muñoz, Newstead, ...)

- **TQFT method** Virtual classes of SL_2-character varieties

  (González-Prieto, Logares, Muñoz, ...)

Today $G = \ldots$

$$\mathbb{T}_n = \text{upper triangular } n \times n \text{ matrices}$$

$$\mathbb{U}_n = \text{unipotent } n \times n \text{ matrices}$$

Results

- **TQFT method** Virtual classes $R_{\mathbb{T}_n}(\Sigma_g)$ for $n = 1, \ldots, 5$
- **Arithmetic method** E-polynomials $R_{\mathbb{U}_n}(\Sigma_g)$ for $n = 1, \ldots, 10$

E-polynomial $$e(X) = \sum_{k,p,q} (-1)^k h_c^{k;p,q}(X)\, u^p v^q \in \mathbb{Z}[u, v]$$

$e(X) = e(Z) + e(X \setminus Z)$ for $Z \subset X$ closed subvariety

$e(X \times Y) = e(X)\, e(Y)$

Grothendieck ring of varieties $\mathsf{K}(\mathbf{Var}_k) = \mathbb{Z}[\mathbf{Var}_k]/\sim$

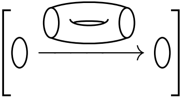
$[X] = [Z] + [X \setminus Z]$ for $Z \subset X$ closed subvariety

$[X \times Y] = [X]\,[Y]$

$$e : \mathsf{K}(\mathbf{Var}_k) \to \mathbb{Z}[u, v]$$

TQFT method

Definition: a **TQFT** is a (lax) monoidal functor $Z : \mathbf{Bord}_n \to R\text{-}\mathbf{Mod}$



$$\overset{Z}{\longmapsto} \quad \left[M \xrightarrow{f} M \right]$$

TQFT method:

$$n = 2 \qquad\qquad R = \mathsf{K}(\mathbf{Var}_k) \qquad\qquad Z(\bigcirc) = \mathsf{K}(\mathbf{Var}/G)$$

$Z($$)(X \to G) =$

$$\begin{bmatrix} X \times G^2 \to G \\ (x, A, B) \mapsto x[A, B] \end{bmatrix}$$

$$\begin{tikzcd}
pt \arrow[r, mapsto] & \{1\} \arrow[r, mapsto] \arrow[d] & G^2 \arrow[r, mapsto] \arrow[d, "{[A_1, B_1]}"] & G^4 \arrow[r, mapsto, "\cdots"] \arrow[d, "{[A_1, B_1][A_2, B_2]}"] & G^{2g} \arrow[r, mapsto] \arrow[d, "{\prod_{i=1}^{g} [A_i, B_i]}"] & R_G(\Sigma_g) \\
& G & G & G & G &
\end{tikzcd}$$

Goal: compute $Z(\;) : \mathsf{K}(\mathbf{Var}/G) \to \mathsf{K}(\mathbf{Var}/G)$

Use *unipotent conjugacy classes* as generators in $\mathsf{K}(\mathbf{Var}/G)$

$$\mathcal{U}_1 \quad \mathcal{U}_2 \quad \dots \quad \mathcal{U}_M$$

Definition: Let G act on X. Then $\xi \in X$ is an *algebraic representative* if exists $\gamma : X \to G$ such that $x = \gamma(x) \cdot \xi$ for all $x \in X$

Fact: Every conjugacy class of $\mathbb{T}_n$ and $\mathbb{U}_n$ has an algebraic representative

Why: If $Y \xrightarrow{f} X$ is G-equivariant, and $\xi \in X$ an algebraic representative

$$\begin{tikzcd} Y \arrow[rr, "y \mapsto (f(y), \gamma(f(y))^{-1} \cdot y)", "\simeq"'] \arrow[dr, "f"'] & & X \times f^{-1}(\xi) \arrow[dl, "\pi_X"] \\ & X & \end{tikzcd}$$

so $[Y] = [X] \cdot [f^{-1}(\xi)]$

$Z($  $)(\{1\})|_{\mathcal{U}_i} = \left[\left\{(A, B) \in G^2 : [A, B] \in \mathcal{U}_i\right\}\right]$

$$= \sum_j [\{(A, B) \in G \times \mathcal{C}_j : [A, B] \in \mathcal{U}_i\}]$$

$$= \sum_j [\{A \in G : [A, \xi_j] \in \mathcal{U}_i\}] \times [\textsf{Orbit}(\xi_j)]$$

$$= \sum_j E_{ij} \, [\textsf{Orbit}(\xi_j)],$$

with $\quad E_{ij} = [\{A \in G : [A, \xi_j] \in \mathcal{U}_i\}]$

$Z($  $)(\mathcal{U}_j)|_{\mathcal{U}_i} = \left[\left\{(g, A, B) \in \mathcal{U}_j \times G^2 : g[A, B] \in \mathcal{U}_i\right\}\right]$

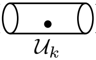
$$= \sum_k \left[\left\{(g, A, B) \in \mathcal{U}_j \times G^2 : g[A, B] \in \mathcal{U}_i, [A, B] \in \mathcal{U}_k\right\}\right]$$

$$= \sum_k [\{g \in \mathcal{U}_j : g\xi_k \in \mathcal{U}_i\}] \cdot \left[\left\{(A, B) \in G^2 : [A, B] \in \mathcal{U}_k\right\}\right]$$

$= \sum_k F_{ijk} \cdot Z($  $)(\{1\})|_{\mathcal{U}_k}$

with $\quad F_{ijk} = [\{g \in \mathcal{U}_j : g\xi_k \in \mathcal{U}_i\}]$

Bonus: automatically version with parabolic data

$Z($ 

$)(\mathcal{U}_j)|_{\mathcal{U}_i} = [\{(g, h) \in \mathcal{U}_j \times \mathcal{U}_k : gh \in \mathcal{U}_i\}] = F_{ijk} \cdot [\mathcal{U}_k]$

(for $G = \mathbb{T}_5$)

	# computations	# variables
naive	$61^2 = 3721$	$15 + 15 = 30$
E_{ij}	$61 \times 372 = 22\,692$	≈ 15
F_{ijk}	$61^3 = 226\,981$	15

61×61 matrix with polynomials of degree 28

$\downarrow$

diagonalize $Z($$) = PDP^{-1}$

$\downarrow$

take powers $Z($$)^g = PD^gP^{-1}$

Final formula

$$
\begin{aligned}
[R_{\mathbb{T}_5}(\Sigma_g)] = {} & q^{12g-2}(q-1)^{6g+2} + 2q^{14g-4}(q-1)^{4g+3} + 3q^{14g-4}(q-1)^{6g+2} + q^{14g-4}(q-1)^{8g+1} \\
& + 2q^{16g-6}(q-1)^{2g+4} + 7q^{16g-6}(q-1)^{4g+3} + 7q^{16g-6}(q-1)^{6g+2} + 2q^{16g-6}(q-1)^{8g+1} \\
& + 2q^{18g-8}(q-1)^{2g+4} + 7q^{18g-8}(q-1)^{4g+3} + 8q^{18g-8}(q-1)^{6g+2} + 3q^{18g-8}(q-1)^{8g+1} \\
& + q^{20g-10}(q-1)^{10g} + q^{20g-10}(q-1)^{2g+4} + 4q^{20g-10}(q-1)^{4g+3} + 6q^{20g-10}(q-1)^{6g+2} \\
& + 4q^{20g-10}(q-1)^{8g+1}
\end{aligned}
$$

Arithmetic method

Katz' theorem: Let X variety over $\mathbb{C}$.
If $\#X(\mathbb{F}_q)$ is polynomial in q,
then $e(X)$ is that polynomial in $q = uv$

Frobenius formula: If G finite group, then

$$\#R_G(\Sigma_g) = \#G \cdot \sum_{\chi \in \mathrm{irr}(G)} \left(\frac{\#G}{\chi(1)} \right)^{2g-2}$$

Conclusion: study representations of G over $\mathbb{F}_q$

Definition: the *representation ζ-function* of G is

$$\zeta_G(s) = \sum_{\chi \in \mathsf{irr}(G)} \chi(1)^{-s},$$

Examples

$$\zeta_{S_3}(s) = 1 + 1 + 2^{-s}$$

$$\zeta_{G\times H}(s) = \zeta_G(s) \cdot \zeta_H(s)$$

$$\zeta_{\mathbb{G}_m(\mathbb{F}_q)}(s) = q - 1$$

$$\zeta_{\mathbb{G}_a(\mathbb{F}_q)}(s) = q$$

$$\boxed{\#R_G(\Sigma_g) = \#G^{2g-1} \cdot \zeta_G(\chi(\Sigma_g))}$$

Theorem

- Let $G = N \rtimes H$ with N abelian
- H acts on the characters $X = \mathrm{Hom}(N, \mathbb{C}^*)$ of N

$$(h \cdot \chi)(n) = \chi(hnh^{-1})$$

- Choose representatives χ_i for every $i \in X/H$
- Let $H_i = \{h \in H : h \cdot \chi_i = \chi_i\}$

Then every irreducible representation of G is of the form

$$\mathrm{Ind}_{N \rtimes H_i}^{G}(\chi_i \otimes \rho) \quad \text{with} \quad \rho \in \mathrm{irr}(H_i)$$

Corollary $$\zeta_{N \rtimes H}(s) = \sum_{i \in X/H} \zeta_{H_i}(s) \cdot [H : H_i]^{-s}$$

Apply to

$$\mathbb{U}_n \qquad = \qquad \mathbb{G}_a^{n-1} \qquad \rtimes \qquad \mathbb{U}_{n-1}$$

$$\left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 & & & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\} \rtimes \left\{ \begin{pmatrix} 1 & * & * & \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}$$

Final formulas $(n = 1, \ldots, 10)$

$$\zeta_{\mathbb{U}_1}(s) = 1$$

$$\zeta_{\mathbb{U}_2}(s) = q$$

$$\zeta_{\mathbb{U}_3}(s) = q^2 + q^{-s}(q-1)$$

$$\zeta_{\mathbb{U}_4}(s) = q^3 + q^{1-2s}(q-1) + q^{1-s}(q-1)(q+1)$$

$$\zeta_{\mathbb{U}_5}(s) = q^4 + q^{1-3s}(q-1)(2q-1) + q^{1-2s}(q-1)(q+1)(2q-1) + q^{2-s}(q-1)(2q+1) + q^{-4s}(q-1)^2$$

$$\zeta_{\mathbb{U}_6}(s) = q^5 + q^{1-6s}(q-1)^2 + q^{1-5s}(q-1)^2(2q+1) + q^{2-3s}(q-1)(q+1)(4q-3) + q^{2-2s}(q-1)(q+2)\left(q^2+q-1\right) + q^{3-s}(q-1)(3q+1) + q^{-4s}(q-1)\left(2q^2-1\right)\left(q^2+q-1\right)$$

$$\zeta_{\mathbb{U}_7}(s) = q^6 + q^{1-8s}(q-1)^2(3q-2) + q^{1-6s}(q-1)^2\left(4q^3+7q^2-3q-1\right) + q^{1-5s}(q-1)\left(q^5+7q^4-2q^3-9q^2+3q+1\right) + q^{1-4s}(q-1)(2q-1)\left(q^4+5q^3-3q-1\right) + q^{2-3s}(q-1)\left(3q^4+6q^3-2q^2-5q+1\right) + q^{3-2s}(q-1)(q+1)\left(2q^2+3q-3\right) + q^{4-s}(q-1)(4q+1) + q^{-7s}(q-1)^2\left(5q^3-3q+1\right) + q^{-9s}(q-1)^3$$

$$\zeta_{\mathbb{U}_8}(s) = q^7 + q^{1-12s}(q-1)^3 + q^{1-11s}(q-1)^3(3q+1) + q^{1-8s}(q-1)^2\left(q^5+13q^4+8q^3-14q^2-4q+3\right) + q^{1-7s}(q-1)^2\left(6q^5+18q^4+4q^3-19q^2+q+3\right) + q^{1-5s}(q-1)\left(2q^7+8q^6+13q^5-23q^4-9q^3+12q^2-1\right) + q^{2-4s}(q-1)\left(12q^5+9q^4-16q^3-9q^2+6q+1\right) + q^{3-6s}(q-1)\left(q^5+7q^4+16q^3-24q^2-14q+15\right) + q^{3-3s}(q-1)\left(q^5+5q^4+10q^3-7q^2-8q+3\right) + q^{4-2s}(q-1)(3q+2)\left(q^2+2q-2\right) + q^{5-s}(q-1)(5q+1) + q^{-9s}(q-1)^2\left(4q^5+10q^4-7q^3-8q^2+3q+1\right) + q^{-10s}(q-1)^2\left(5q^4+q^3-6q^2+1\right)$$

Comparison

TQFT method	Arithmetic method
Virtual class $\mathsf{K}(\mathbf{Var}_k)$	E-polynomial $\mathbb{Z}[u, v]$
Complexity grows quickly!	Managable
$1 \leq n \leq 5$	$1 \leq n \leq 10$
'Geometric insight'	Specific case